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Are Kaluza–Klein Models of the Universe Chaotic?

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The generic behavior of vacuum inhomogeneous and spatially homogeneous Kaluza–Klein models is studied in the vicinity of the cosmological singularity. It is shown that, in space-time dimensions ≥ 11, the generalized Kasner solution, with monotonic power-law behavior of the spatial distances, becomes a general solution of the Einstein vacuum field equations and that, moreover, the chaotic oscillatory behavior disappears.

On the other hand, the chaotic oscillatory behavior, absent in diagonal spatially homogeneous cosmological models in space-time dimensions between 5 and 10, can be reestablished when off-diagonal terms are included.

1. INTRODUCTION

Recent interest in the unification of the fundamental interactions has revived interest in gravitational theories in higher-dimensional space-times, in the line of the old 5-dimensional Kaluza–Klein theory (Kaluza, 1921; Klein, 1926). Our usual $3+1$ space-time does not seem big enough to accommodate today's gauge theories of particle physics: ten or eleven dimensions are required by superstring and supergravity theories, respectively. In this framework, a new domain of cosmology has developed: the study of the dynamics of multidimensional cosmological models.

Adopting this point of view and assuming accordingly that the space-time of our universe can be described by a pseudo-Riemannian manifold of dimension $d+1$, with a metric satisfying Einstein's field equations (which is a minimal assumption of most of the recently proposed unified theories), we are led to ask the following question: are the fundamental geometric

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properties of general relativistic four-dimensional cosmological models conserved for any value of the space dimension d?

In particular, it is well known that one of the outstanding results in four-dimensional theoretical cosmology, due to Belinskii *et al.* (1970),[5] is that the general solution to the gravitational field equations in the vicinity of the initial cosmological singularity is an inhomogeneous generalization of the "Mixmaster universe." The Mixmaster universe (Misner, 1969) is a spatially homogeneous diagonal model whose homogeneity group is $SO(3)$ and which exhibits chaotic properties. Besides this generic solution, a similar inhomogeneous generalization of the well-known Kasner solution contains one arbitrary function less than the generalized Mixmaster metric (Khalatnikov and Lifshitz, 1963). This latter solution is nonchaotic.

Do these four-dimensional qualitative features of a generic solution near the initial singularity remain valid in higher-dimensional Kaluza–Klein cosmologies? This question is far from trivial, in the sense that it has been repeatedly observed, both in supergravity and superstring models, that particular properties true in particular space-time dimensions do not necessarily hold in other dimensions. Moreover, this problem is of great importance in the context of the spontaneous compactification hypothesis, since the alleged decoupling between the ordinary expanding 3-space and a $(d-3)$ compact manifold of a size of the order of Planck's length could be hindered by the eventual presence of chaos in the generic solutions of $(d+1)$-dimensional Einstein field equations.

In the following sections, we describe recent work settling the question of the general behavior of vacuum inhomogeneous as well as spatially homogeneous multidimensional cosmological models in the neighborhood of the initial cosmological singularity.

2. EXISTENCE OF A CRITICAL SPACE-TIME DIMENSION FOR THE GENERAL BEHAVIOR OF VACUUM INHOMOGENEOUS KALUZA-KLEIN COSMOLOGIES

As a first result, we have shown that, contrary to previous expectations, *the generalized Kasner solution*, with monotonic power-law behavior of the spatial distances, *becomes a general solution of Einstein's field equations near the cosmological singularity in space-time dimensions* ≥ 11 (Demaret *et al.*, 1985*a*).

The generalized Kasner solutions reads

$$ds^2 = -dt^2 + \sum_{i=1}^{d} t^{2p_i(x)}(\omega^i)^2 \tag{1}$$

[5]For a controversial discussion of the reality of the alleged generality of this solution see Barrow and Tipler (1979, 1981), Belinskii *et al.* (1980), and ter Haar (1981).

where d is the number of spatial dimensions and $p_i(x)$ are d time-independent functions of the spatial coordinates submitted to the Kasner conditions

$$\sum_{i=1}^{d} p_i(x) = 1 = \sum_{i=1}^{d} p_i^2(x) \tag{2}$$

where the time-independent forms ω^i are arbitrary.

In the vicinity of the initial singularity ($t \to 0$), the time derivative terms in the vacuum Einstein equation ${}^{(d+1)}G_0^0$ and ${}^{(d+1)}G_b^a$ are with (1) potentially of order t^{-2}, just as in four dimensions (Khalatnikov and Lifshitz, 1963). In order for the metric (1) to satisfy these field equations, the spatial gradients present in the spatial curvature term should be negligible, which is possible only if

$$\lim_{t\to 0} t^2 \, {}^{(d)}R_b^a = 0 \tag{3}$$

where ${}^{(d)}R_b^a$ is the d-dimensional spatial Ricci tensor.

The leading terms of $t^2 \, {}^{(d)}R_b^a$ contain the powers $t^{2\alpha_{ijk}}$, where the exponents α_{ijk} are given by

$$\alpha_{ijk} = 2p_i + \sum_{l \neq i,j,k} p_l \tag{4}$$

with $i \neq j$, $i \neq k$, $j \neq k$.

If the number of spatial dimensions is less than 10, it can be shown that there does not exist any *open* region of the "Kasner sphere" defined by (2), where all exponents α_{ijk} would be positive (Demaret *et al.*, 1985*a*). On the other hand, for $d \geq 10$, these exponents are all strictly positive in a suitable neighborhood of $p_1 = p_2 = p_3 = (1 - \sqrt{21})/10$ and $p_4 = p_5 = \cdots = p_{10} = (7 + 3\sqrt{21})/70$. The existence of negative exponents α_{ijk}, leading to divergent terms in the field equations, restricts the generality of the generalized Kasner solution by requiring the introduction of "extra" conditions on the components of the 1-forms ω^i and on their derivatives: these conditions express that the coefficients of $t^{2\alpha_{ijk}}$ with negative α_{ijk} in $t^2 \, {}^{(d)}R_b^a$ are absent. This is what happens in four space-time dimensions ($d = 3$) as well as for $4 < d < 10$.

On the other hand, if the exponents α_{ijk} are all positive, it is easily checked that the $d(d+1)$ arbitrary functions appearing in the metric (1) are linked by d constraints, i.e., the ${}^{(d+1)}G_a^0$ equations and by the two relations (2) defining the Kasner $(d-2)$-sphere. If we also take into account the possibility of fixing d functions of the coordinates characterizing the residual invariance of (1), i.e., spatial reparametrization, we are left with $d(d+1) - d - 2 - d = (d+1)(d-2)$ physically distinct arbitrary functions in (1): this number is equal to twice the number of degrees of freedom of the gravitational field (spin 2-field) in $(d+1)$ spacetime dimensions. *The Kasner generalized metric* (1) *becomes, then, when* $d \geq 10$, *a general solution of the vacuum Einstein equations, in the neighborhood of the singularity.* Incidentally, this result indicates that a general solution to the vacuum gravitational field equations in $d+1$ dimensions cannot necessarily be interpreted in four dimensions as describing the coupled Einstein–Yang–Mills-Scalar system, for which it is known that the oscillatory behavior exists (Belinskii and Khalatnikov, 1973).

3. DISAPPEARANCE OF THE CHAOTIC OSCILLATION BEHAVIOR FOR VACUUM INHOMOGENEOUS KALUZA–KLEIN COSMOLOGICAL MODELS IN SPATIAL DIMENSIONS $d \geq 10$.

The question is now whether, besides the generalized Kasner solution, the oscillatory "mixmaster" behavior found by Belinskii *et al* (1970) and Misner (1969) remains stable for $d \geq 10$. It would not be so if initial data of the Mixmaster type led, after a finite member of curvature-induced collisions, to Kasner exponents which belong to the Kasner stability region.

Initial data chosen at random will of course not ensure the positivity of all the exponents (4). Then, the influence of some spatial curvature terms grows as one goes toward the singularity. These terms cannot be neglected and eventually modify the metric, giving rise to a succession of Kasnerian regimes connected by a collision law which can be described as follows. If the exponents $p_1, \ldots, p_d$ (arranged in increasing order) define on the Kasner sphere (2) a point outside the stability region $\alpha_{ijk} > 0$, $\forall_{i,j,k}$, then these exponents will transform, for $t \to 0$, into a set of d new Kasner exponents $p'_1, \ldots, p'_d$ given by

$$p'_1, p'_2, \ldots, p'_d = \text{ordering of } (q_1, \ldots, q_d) \tag{5}$$

with

$$\begin{aligned} q_1 &= -\pi^{-1}(p_1 + p) \\ q_2 &= -\pi^{-1} p_2 \\ &\vdots \\ q_{d-2} &= \pi^{-1} p_{d-2} \\ q_{d-1} &= \pi^{-1}(2p_1 + p_{d-1} + P) \\ q_d &= \pi^{-1}(2p_1 + p_d + P) \end{aligned} \tag{6}$$

where

$$P = \sum_{i=2}^{d-2} p_i$$

and

$$\pi = 1 + 2p_1 + P \tag{7}$$

The q_i given by (6) are in general not in increasing order. This is why it is necessary to reorder them in order to get the new Kasner exponents, p_i'. It is this rearrangement which makes the transformation (5)–(7) non-trivial.

Now, if the p_i' are such that all the new α'_{ijk} are positive, the oscillatory regime ceases. If, on the contrary, some α'_{ijk} are negative, then the "collision process" goes on until one reaches Kasner exponents which belong to the Kasner stability region. So, the question raised above becomes: Does repeated application of the collision law (5), (6) map almost all initial data into the region $\alpha_{ijk} > 0$?

From a theoretical study and a numerical analysis of the properties of the transformation (5)–(7) carried for $d = 10$, we have argued that the Kasner stability region is the most likely target for this mapping (Demaret *et al.*, 1986). This numerical study, based on 10^5 initial data p_i chosen at random on the Kasner sphere, has led, for each set of p_i, to a final situation free from oscillations (Demaret *et al.*, 1986).

A full theoretical proof of the permanence of chaos for space-time dimensions ≤ 10 and its disappearance for $d+1 \geq 11$ has, however, recently been constructed (Elskens and Henneaux, 1987*a,b*; Elskens, 1987). One of the main ingredients of this proof, which makes it successful, is the introduction, for any space-time dimension, of a new parametrization of the Kasner exponents which reduces the Mixmaster dynamics to a combination of a translation and an isometry or dilating inversion. In particular, in the case of five space-time dimensions, a Markov partition has been constructed and a generalized K-property proven, while for $d \leq 9$, it has been possible to show that the Mixmaster map is ergodic and topologically mixing. By constrast, for $d \geq 10$, the Mixmaster map reduces to the identity after a finite number of iterations, except for a set of initial data with zero Lebesgue measure.

On the basis of these results, it is clear that *the chaotic oscillatory behavior is no longer generic for* $d \geq 10$, and that *this regime is replaced by the monotonic generalized behavior*, which should make the dynamical scenario of dimensional reduction considered by Chodos and Detweiler (1980) more plausible, in this last case. At the same time, it appears that Kasner exponents can be used to characterize spacelike singularities for

$d \geq 10$, and could therefore yield appropriate quantum numbers to specify the "in-states" coming out a singularity (Henneaux *et al.*, 1982).

The validity of this conclusion has been confirmed by a Hamiltonian analysis, in the framework of the ADM formalism for general relativity, of the approach to a space-time singularity of a general inhomogeneous universe in an arbitrary number of dimensions (Hosoya *et al.*, 1987; Jantzen, 1987).

4. THE CASE OF KALUZA–KLEIN SPATIALLY HOMOGENEOUS COSMOLOGICAL MODELS

Spatially homogeneous cosmological models (including Bianchi as well as Friedmann–Robertson–Walker models) play a major role in today's research in theoretical cosmology. It is thus of direct interest to investigate whether the preceding conclusions concerning the occurrence of chaos in multidimensional inhomogeneous models hold in simpler homogeneous models.

The beginning of an answer to this question has been given in particular cases (Furusawa and Hosoya, 1985; Barrow and Stein–Schabes, 1985; Tomimatsu and Ishihara, 1986; Ishihara, 1985; Halpern, 1986; Demianski *et al.*, 1986, 1987), where no chaos was found in some diagonal solutions. This result has been generalized in Demaret *et al.* (1986), where it was explained that homogeneous solutions which are diagonal in the canonical basis do not fulfil the criterion necessary to generate the oscillatory behavior and hence are unable to mimic the full complexity of inhomogeneous cosmologies. This has been explicitly confirmed in Jensen (1987). Hence, it appears that if one insists on diagonal models, inhomogeneities are necessary to trigger the oscillations for spacetime dimensions $d+1$, with $4 \leq d \leq 9$, and thus play a more important role for $d > 3$ than they do for $d = 3$.

These results, however interesting they may be, nevertheless open the question of the behavior of non-diagonal homogeneous solutions. Since diagonal models are extremely peculiar for $d > 3$ and generically correspond to initial data forming a set of measure zero (Jantzen, 1986, 1987), the answer to this question is not entirely immediate.

The main result that we have obtained is that *the inclusion of nondiagonal terms in spatially homogeneous models restores the oscillatory behavior*, provided the homogeneity group is not trivial (Demaret *et al.*, 1988).

The homogeneity assumption implies that the metric is given, for this class of models, by

$$ds^2 = -dt^2 + g_{ij}(t)\mathring{\omega}^i\mathring{\omega}^j \tag{8}$$

where the $\mathring{\omega}^i$ are time-independent canonical bases invariant under the group transformations,

$$d\mathring{\omega}^i = -\tfrac{1}{2}\mathring{C}^i_{jk}\mathring{\omega}^j \wedge \mathring{\omega}^k \tag{9}$$

and where the metric $g_{ij}(t)$ depends only on time and may contain off-diagonal terms (Jantzen, 1986, 1987; Ryan and Shepley, 1975; Demaret *et al.*, 1985).

The Einstein dynamical equations are first-order differential equations with respect to time for the metric $g_{ij}(t)$ and the extrinsic curvature $K_{ij}(t)$,

$$\begin{aligned} g'_{ij} &= -2K_{ij} \\ (\sqrt{g}\,K^j_i)'/\sqrt{g} - P^j_i &= 0 \end{aligned} \tag{10}$$

Here, P^j_i is the spatial curvature of the d-dimensional metric $g_{ij}\mathring{\omega}^i\mathring{\omega}^j$

$$P_{ij} = -\sigma^c_{id}\sigma^d_{jc} - \mathring{C}^d_{dc}\sigma^c_{ij} \tag{11}$$

$$\sigma^c_{ij} = \tfrac{1}{2}(\mathring{C}^c_{ij} + \mathring{C}^e_{jd}g_{ei}g^{dc} - \mathring{C}^e_{di}g_{ej}g^{dc}) \tag{12}$$

and the spatial indices are raised or lowered with g_{ij}.

The general solution of (10) is completely determined by $d(d+1)$ initial data at a given "initial" time t_0. These initial data are not completely arbitrary, since they should obey the constraint equations

$$(K_{ij}K^{ij} - K^2) - P = 0 \qquad (P \equiv P^i_i) \tag{13}$$

$$g^{mn}K_{nr}(\mathring{C}^r_{mi} - \delta^r_i\mathring{C}^t_{tm}) = 0 \tag{14}$$

The number of independent equations contained in (14) depends on the homogeneity group under consideration.

In order to investigate the qualitative properties of the solutions of (8), it is convenient to redefine the dynamical variables as follows. By a linear transformation of the invariant bases $\mathring{\omega}^i$, one can simultaneously diagonalize g_{ij} and K_{ij},

$$ds^2 = -dt^2 + \sum_{i=1}^{d} a_i^2(t)(\omega^i)^2 \tag{15}$$

$$K_{ij}\mathring{\omega}^i\mathring{\omega}^j = \sum_{i=1}^{d} k_i(t)(\omega^i)^2 \tag{16}$$

$$\omega^i = \Lambda^i_j(t)\mathring{\omega}^j \tag{17}$$

The axes which achieve this diagonalization are generically time dependent and will be referred to as the "Kasner axes." The frame ω^i is determined up to rescalings of the distances along the axes. We will choose a time-independent scale, i.e., we will impose the conditions

$$(\Lambda\Lambda^t)^i_i = 1 \tag{18}$$

(for each i), as in Belinskii *et al.* (1972).

The formulas (15)-(18) define an invertible change of variables from the $d(d+1)$ functions g_{ij}, K_{ij} to the $d(d+1)$ functions a_i, k_i, and Λ^i_j. The new variables turn out to possess a simpler qualitative behavior.

The analysis of the solutions of (10) starts by considering an epoch where the spatial gradients are dominated by the time derivatives in the Einstein equations, i.e. (Belinskii *et al.*, 1972), at the initial time,

$$P_{ij} \ll t^{-2}(g_{ii})^{1/2}(g_{jj})^{1/2} \tag{19}$$

When the inequalities (19) hold, the integration of equations (10) yields the Kasner solutions

$$\Lambda^i_j = \mathring{\Lambda}^i_j \tag{20}$$

$$a_i(t) = \mathring{a}_i t^{p_i} \tag{21}$$

where $\mathring{\Lambda}^i_j$, $\mathring{a}_i$, and p_i are integration constants. The p_i are called Kasner exponents and are subject to

$$\sum_{i=1}^{d} p_i = \sum_{i=1}^{d} p_i^2 = 1 \tag{22}$$

The Kasner solution (20)-(21) holds as long as the condition (19) is fulfilled. Now, one sees from (11), (12), (15)-(18), and (20)-(21) that in the Kasner frame ω^i, the diagonal components $t^2P^i_i$ of the spatial curvature involve the terms

$$A_{ijk} \equiv a_i^2 \prod_{s\neq i, s\neq j, s\neq k} a_s \qquad (i\neq j, \quad i\neq k, \quad j\neq k) \tag{23}$$

Furthermore, these terms *all* appear in $t^2P^i_i$ if none of the structure constants C^i_{jk} with three different indices ($i\neq j$, $i\neq k$, $j\neq k$) vanishes in the Kasner basis.

As we will show below, this latter condition, which does not hold in the canonical bases $\mathring{\omega}^i$ when $d\geq 4$ (Demaret *et al.*, 1986; Jensen, 1987) is, however, generically verified in noncanonical bases for interesting groups, even when $d\geq 4$. So, let us assume $C^i_{jk}\neq 0$ ($i\neq j$, $i\neq k$, $j\neq k$).

The conditions (19) with $i=j$ are then equivalent to

$$A_{ijk}k/\Lambda \ll 1 \tag{24}$$

where $1/k$ denotes the order of magnitude of the distances (determined by C^i_{jk}) over which the metric varies significantly in a coordinate basis, and where Λ is the product $\mathring{a}_1\mathring{a}_2 \ldots \mathring{a}_d$, so that

$$\sqrt{g} = \Lambda t \tag{25}$$

If one replaces a_i by t^{p_i} in A_{ijk}, one finds

$$A_{ijk} = t^{\alpha ijk} \tag{26}$$

$$\alpha_{ijk} = 2p_i + \sum_{s \neq i, s \neq j, s \neq k} p_s \qquad (i \neq j, \quad i \neq k, \quad j \neq k) \tag{27}$$

As shown in Demaret *et al.* (1985a) and recalled above, the exponents α_{ijk} can all be taken strictly positive for $d \geq 10$ [so that $A_{ijk} \to 0$ as $t \to 0$ and (23) remains fulfilled], while for $d \leq 9$, at least one a_{ijk} is negative. If the Kasner exponents are ordered, $p_1 \leq p_2 \leq p_3 \cdots \leq p_d$, one then has

$$\alpha_{1d-1d} < 0 \tag{28}$$

This implies that the condition necessary for the validity of the Kasner solution cannot be fulfilled permanently as $t \to 0$ and that the Kasner solution will be replaced by another one as one approaches the singularity (Belinskii *et al.*, 1970; Demaret *et al.*, 1986).

This new solution is obtained by integrating the Einstein equations, keeping the nonnegligible spatial curvature terms. If one disregards the case of small oscillations—which does not appear to change qualitatively the conclusions (Belinskii *et al.*, 1970)—the dominant nonnegligible term is A_{1d-1d}, i.e., when

$$A_{1d-1d}k/\Lambda \simeq 1 \tag{29}$$

All other A_{ijk} still obey (24). In that case, only A_{1d-1d} needs to be retained in the diagonal Einstein equations. The condition of applicabliity of this approximation is clearly

$$a_1 \gg a_2 \gg a_3 \cdots a_{d-2} \gg a_{d-1} \gg a_d \tag{30}$$

and it will be assumed throughout (case of "nonsmall oscillations").

It is easy to check that the terms present in the off-diagonal components of the spatial Ricci tensor are all negligible compared with A_{1d-1d}, so that the off-diagonal equations remain satisfied to zeroth order, even when A_{1d-1d} is not small in the sense of (29).

Thus, to that order, the change in the Kasner solution is induced by the sole diagonal equations, which replace the Kasner exponents p_i by new ones p'_j according to the rule (5)-(7) (Belinskii and Khalatnikov, 1973; Demaret *et al.*, 1986).

Now, the crucial point is that another collision will follow (5)-(7) and the same scenario will repeat itself indefinitely as $t \to 0$. This is because $C^i_{jk} \neq 0$ ($i \neq j$, $i \neq k$, $j \neq k$) in the Kasner basis ω^i, so that it is guaranteed that a final Kasner regime can never settle down since the validity conditions (19) for such a regime always ultimately get violated by at least one term.

One thus sees that the qualitative behavior of a homogeneous cosmological solution (with $C^i_{jk} \neq 0$, $i \neq j$, $i \neq k$, $j \neq k$) is characterized by an infinite series of collisions according to (5)-(7), which is known to exhibit interesting chaotic properties (Elskens and Henneaux, 1987*a*,*b*; Elskens, 1987). The conclusion is therefore that *spatially homogeneous cosmological models which are not diagonal in the canonical basis (so as to have $C^i_{jk} \neq 0$ for different indices in the Kasner basis) do possess the interesting features of their inhomogeneous generalizations.*

As a final point, we have investigated in more detail the conditions under which $C^i_{jk} \neq 0$ ($i \neq j$, $i \neq k$, $j \neq k$) in a generic noncanonical basis. There are two features which could force $C^i_{jk} \neq 0$ ($i \neq j$, $i \neq k$, $j \neq k$) to vanish in any frame:

(a) The group is too "trivial" (e.g., in the Abelian case, C^i_{jk} clearly vanish in all invariant frames).
(b) The spatial constraints (14) could restrict Λ^i_j in such a way that one C^i_{jk} is zero for all triples of different indices. This second possibility arises, for instance, for Bianchi types II, IV, VI, or VII in $(3+1)$ dimensions.

However, we have explicitly found spatial homogeneity groups such that C^i_{jk} ($i \neq j$, $i \neq k$, $j \neq k$) differs from zero in a generic noncanonical basis, compatible with the spatial constraints. These are class A groups ($C^i_{ji} = 0$) of the type G_{13}[$(4+1)$ dimensions (Fee, 1979)], $G_{13} \otimes U_1$ [$(5+1)$ dimensions], $SO(3) \otimes SO(3)$ [$(6+1)$ dimensions], $SO(3) \otimes SO(3) \otimes U(1)$ [$(7+1)$ dimensions], $SU(3)$ [$(8+1)$ dimensions], or $SU(3) \otimes U(1)$ [$(9+1)$ dimensions]. By a linear transformation of the structure constants, one can find a frame where $C^i_{jk} \neq 0$ ($i \neq j$, $i \neq k$, $j \neq k$) and where the constraints (14) are satisfied. In such a frame, the condition for an infinite number of collisions is satisfied.

As an explicit example of a spatially homogeneous model with $d \leq 9$ which exhibits the chaotic behavior, we consider a model in 8 spatial dimensions, invariant with respect to the group of spatial homogeneity $SU(3)$ (de Rop, 1988). If the transformation between the canonical vector basis $\mathring{X}_i$ and the general basis X_i is chosen as infinitesimal, i.e.,

$$X_i = (\delta^j_i - \varepsilon^j_i)\mathring{X}_j \tag{31}$$

where ε^j_i are infinitesimal constants, the modification of the canonical structure constants $\mathring{C}^i_{jk}$, is given, at first order in ε, by

$$\delta C^i_{jk} = \varepsilon^i_m \mathring{C}^m_{jk} - \varepsilon^m_j \mathring{C}^i_{mk} - \varepsilon^m_k \mathring{C}^i_{jm} \tag{32}$$

Moreover, the constraint equations (14) can be written, in the case of a class A group ($\mathring{C}^i_{ji} = 0$) such as $SU(3)$, as

$$\sum_k p_k \delta C^k_{ik} = 0 \tag{33}$$

If one chooses now the generators of $SU(3)$ such that the group metric tensor be a multiple of unity, which ensures that the $\mathring{C}^i_{jk}$ will be completely antisymmetric (Gilmore, 1974), the constraints (33) become, after transformation (31),

$$\begin{aligned}
&2(\varepsilon^3_2+\varepsilon^2_3)(p_3-p_2)-(\varepsilon^7_4+\varepsilon^4_7)(p_4-p_7)+(\varepsilon^6_5+\varepsilon^5_6)(p_5-p_6)=0\\
&-2(\varepsilon^3_1+\varepsilon^1_3)(p_1-p_3)+(\varepsilon^6_4+\varepsilon^4_6)(p_4-p_6)+(\varepsilon^7_5+\varepsilon^5_7)(p_5-p_7)=0\\
&2(\varepsilon^2_1+\varepsilon^1_2)(p_1-p_2)+(\varepsilon^5_4+\varepsilon^4_5)(p_4-p_5)-(\varepsilon^7_6+\varepsilon^6_7)(p_6-p_7)=0\\
&(\varepsilon^7_1+\varepsilon^1_7)(p_1-p_7)+(\varepsilon^6_2+\varepsilon^2_6)(p_2-p_6)\\
&\qquad+(\varepsilon^5_3+\varepsilon^3_5)(p_3-p_5)-\sqrt{3}(\varepsilon^8_5+\varepsilon^5_8)(p_5-p_8)=0\\
&(\varepsilon^6_1-\varepsilon^1_6)(p_1-p_6)-(\varepsilon^7_2+\varepsilon^2_7)(p_2-p_7)\\
&\qquad+(\varepsilon^4_3+\varepsilon^3_4)(p_3-p_4)+\sqrt{3}(\varepsilon^4_8+\varepsilon^8_4)(p_8-p_4)=0\\
&-(\varepsilon^5_1+\varepsilon^1_5)(p_1-p_5)+(\varepsilon^4_2+\varepsilon^2_4)(p_2-p_4)\\
&\qquad+(\varepsilon^7_3+\varepsilon^3_7)(p_3-p_7)-\sqrt{3}\,(\varepsilon^7_8+\varepsilon^8_7)(p_8-p_7)=0\\
&(\varepsilon^4_1+\varepsilon^1_4)(p_1-p_4)+(\varepsilon^5_2+\varepsilon^2_5)(p_2-p_5)\\
&\qquad-(\varepsilon^6_3+\varepsilon^3_6)(p_3-p_6)-\sqrt{3}(\varepsilon^8_6+\varepsilon^6_8)(p_6-p_8)=0\\
&(\varepsilon^5_4+\varepsilon^4_5)(p_4-p_5)+(\varepsilon^7_6+\varepsilon^6_7)(p_6-p_7)=0
\end{aligned}\tag{34}$$

These relations are satisfied if, e.g., all combinations $(\varepsilon^j_i+\varepsilon^i_j)$ appearing in (34), i.e., the δC^k_{ik} (without summation), $\forall i$, k, are made vanishing. On the other hand, it can be easily checked that the δC^i_{jk} $(i \neq j,\ i \neq k,\ j \neq k)$ are all different from zero and do not contain any term of the form $(\varepsilon^j_i+\varepsilon^i_j)$. This is sufficient to show that, despite the constraints (34), the coefficients of the "dangerous" terms in P_{ij} (i.e., terms which could diverge at least as fast as t^{-2} as $t \to 0$) all vanish for a general basis. We are thus led to the conclusion that *the vacuum nondiagonal spatially homogeneous model, invariant with respect to $SU(3)$, is chaotic near the cosmological singularity.*

In fact, the general result reported here is similar to the analysis of the coupled scalar Maxwell-Einstein system in $(3+1)$ dimensions (Belinskii and Khalatnikov, 1973). When the Maxwell field is set equal to zero, the chaotic behavior disappears. But if a nontrivial Maxwell field is included, the oscillatory regime is restored. Now, an electromagnetic field can be viewed, along Kaluza-Klein lines, as nondiagonal metric components in higher dimensions. This therefore confirms our results.

5. CONCLUSIONS

The results described here have, in our opinion, settled the question of the type of behavior of general vacuum inhomogeneous and spatially homogeneous cosmological models in the neighborhood of the initial singularity, at least in the framework of the approach of the Russian school (Belinskii *et al.*, 1970, 1972; Khalatnikov and Lifshitz, 1963; Belinskii and Khalatnikov, 1973).

Extensions of this work should be carried out in two directions: a detailed study of the role played by the matter fields and the consideration of higher-order gravitational Lagrangians, as required, e.g., by superstring theory. Partial results have recently been obtained on these problems: the influence of the dilaton field and of the three-index tensor field, H on the very early behavior of a generalized Friedmann universe has been studied in the context of the ten-dimensional superstring theory (Henriques and Moorhouse, 1987; Liddle *et al.*, 1989), while it has been shown that the chaotic behavior characteristics of the Mixmaster model in four dimensions disappears in a gravity theory derived from a purely quadratic gravitational Lagrangian, giving rise to the monotonic Kasner-like behavior (Barrow and Sirousse, 1988).

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